

Pairwise Stability in Two Sided Market with Strictly Increasing Valuation Functions

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Abstract

This paper deals with two-sided matching market with two disjoint sets, i.e. the set of buyers and the set of sellers. Each seller can trade with at most with one buyer and vice versa. Money is transferred from sellers to buyers for an indivisible goods that buyers own. Valuation functions, for participants of both sides, are represented by strictly increasing functions with money considered as discrete variable. An algorithm is devised to prove the existence of stability for this model.

keyword: Stable matching, marriage model, indivisible goods, increasing valuations

1 Introduction

Over the last few decades, numerous scholars have carried out research pertaining to two-sided matching problem. In a two sided matching problem, the set of participants are divided into two disjoint sets, say U and V . Each participant ranks a participant of other set in order of preferences. Main objective of two-sided matching problem is formation of partnership between the participants of U and V . A matching X , is one-to-one correspondence between the participant of one set to the participant of other set. Main requirement in a two-sided matching problems is that of stability of matchings. A matching is stable if all participants have acceptable partners and there does not exist a pair that is not matched but prefers each other to their current partners.

The concept of finding two-sided stable matching was first given by Gale and Shapley [5] in their paper “*College Admissions and the Stability of Marriage*”. In the course of presenting an algorithm for matching applicants to college places, they introduced and solved the stable marriage problem. This problem deals with two disjoint sets of participants U and V . Each participant of these sets submits a preference list ranking a subset of other set of participants in order of preference. The aim is to form a one-to-one matching X of the participants such that no two participants would prefer each other to their partner in X . The authors used their solution to this problem as a basis for solving the extended problem where one of the sets consists of college applicants, and the other consists of colleges, each of which has a quota of places to fill. An important feature of their model is that no negotiations are allowed among the participant of both sets. This shows that participants in their model are rigid. Many additional variants of the stable marriage problem have been discussed in the literature. Guessfield and Irving [6] published a book that covers many variants of original

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stable marriage problem such as the preferences of agents may include ties, incomplete preferences, weighted edges as well as non-bipartite versions such as roommate problem.

Shapley and Shubik [8] presented the one-to-one buyer seller model known as “*assignment game*”. In their model, participants are flexible because monetary transfer is permitted among participants of both sets. Each participant on one side can supply exactly one unit of some indivisible good and exchange it for money with a participant from the other side whose demand is also one unit. Shapley and Shubik [8] showed that the core of the game is a non-empty complete lattice, where the core is defined as the set of un-denominated outcomes. The core in their model is a solution set based upon a linear programming formulation of the model [8].

After this, two-sided matchings have been studied extensively. Different approaches have been made by many researchers in which they generalize the marriage model of Gale and Shapley [5] and assignment game of Shapley and Shubik [8]. Main aim of these researchers was to find common result for both of [5] and [8] models in a more general way. Erikson and Karlinder [2] and Sotomayor [9] presented the hybrid models. These models are the generalization of the discrete marriage model [5] and continuous assignment game [8]. Existence of stable outcome and the core is discussed in [2, 9]. Farooq [3] presented a one-to-one matching model in which he identified the preferences of participants by strictly increasing linear functions. He proposed an algorithm to show the existence of pairwise stable outcome in his model by taking money as a continuous variable. His model includes the marriage model of Gale and Shapley [5], assignment game of Shapley and Shubik [8] and Erikson and Karlinder [2] hybrid model as special cases. The motivation of our work from the stable matching literature is the model of Ali and Farooq [1]. Ali and Farooq [1] presented a one-to-one matching model by taking money as a discrete variable in linear increasing function. They designed an algorithm to show that pairwise stable outcome always exists. The complexity of Ali and Farooq’s [1] algorithm depends on the size of those intervals where prices fall. Our model is the generalized form of Ali and Farooq [1] model. We consider the preferences of participants by general increasing function and designed an algorithm to find a pairwise stable outcome in our model.

This paper is organized as follows. Section 2 describes of our model briefly. Section 3 gives the Sequential Mechanisms for buyer and seller. Section 4 describes the supply and demand characterization of stable matching. We devise an algorithm which finds a stable outcome in our model in Section 5. In Section 6, we discuss the main result of our model.

2 The Model Description

The matching market under consideration consists of two types of participants one type of participants are sellers and second type of participants are buyers. Here U and V denote the sets of sellers and buyers, respectively. Throughout in this paper, we model matching markets as trading platforms where buyers and sellers interact. Moreover, each buyer as well as seller can trade with at most one participant on the other side of the market at a particular time. The negotiation and side payments between participants of both sides are allowed. Naturally, each participant wants to gain as much profit as possible from his/her partner. Let $E = U \times V$ denotes the set of all possible pairs of seller-buyer. Also when buyer and seller interact with each other in auction market they have some upper and lower bounds of prices. We express these bounds by vector $\underline{\pi}, \bar{\pi} \in \mathbb{Z}^E$ where always

$\underline{p}_{ij} \leq \bar{p}_{ij}$ for each $(i, j) \in E^2$. The price vector is denoted by p and define as $p = (p_{ij} \in \mathbb{Z} | (i, j) \in E)$. The price vector is said to be feasible price vector if it satisfies $\underline{p} \leq p \leq \bar{p}$ ³.

Since each participant has preferences over the participants of the other set, so the preferences of sellers over buyers and buyers over sellers is given by the valuation function $f_{ij}(x)$ and $f_{ji}(-x)$ for each $(i, j) \in E$. Here $f_{ij}(x)$ denotes the valuation of seller $i \in U$, when he or she trade with buyer $j \in V$, and get an amount of money x from buyer $j \in V$. Similarly, $f_{ji}(-x)$ represents the valuation of buyer $j \in V$, when he or she trade with seller $i \in U$, and pays an amount of money x . These valuation functions are strictly increasing functions of money⁴.

3 The Buyer Seller Sequential Mechanism

Since $f_{ij}(x)$ and $f_{ji}(-x)$ denote the preferences of participants so if $f_{ij}(x) \geq 0$, then we say that seller i is ready to make a partnership with buyer j if j pays i an amount x of money. Or we can say that buyer j is acceptable to seller i an amount x of money. Also if $f_{ji}(-x) \geq 0$, then we say that buyer j is ready to make a partnership with seller i an amount x of money. If $f_{i_0j_0}(x_1) > f_{i_0j_1}(x_1)$, then we can say that seller i_0 *prefers* buyer j_0 to buyer j_1 at money x_1 where $i_0 \in U$ and $j_0, j_1 \in V$ and $x_1 \in \mathbb{Z}$. If $f_{j_0i_0}(-x_1) > f_{j_0i_1}(-x_1)$, then we can say that j_0 *prefers* i_0 to i_1 at money x_1 where $i_0, i_1 \in U$ and $j_0 \in V$ and $x_1 \in \mathbb{Z}$. If $f_{i_0j_0}(x_1) = f_{i_0j_1}(x_1)$, then seller i_0 is *indifferent* between j_0 and j_1 at money x_1 . Also, if $f_{j_0i_0}(-x_1) = f_{j_0i_1}(-x_1)$, then buyer j_0 is said to be *indifferent* between i_0 and i_1 at money x_1 . If $f_{ij}(x) = 0$, then seller i is indifferent between the buyer j and himself at x . If an individual is not indifferent between any two participants then the preferences of such individual are called *strict preferences*. In our model, preferences of the participants are not strict because these are based on monetary transfer and therefore, different functions may have same value for two distinct values of money. If $f_{ji}(-x) = 0$ for some $x \in \mathbb{Z}$, then buyer j is indifferent between the seller i and himself at x . Preferences of participants are not strict in our model because the monetary transfer is allowed between participants of both sets.

4 The Supply and Demand Characterization of Stable Matchings

This section describes the characteristic of an outcome for which it would be stable. A subset X , of a set E , is called matching if every agent appear at most once in X . A matching X is said to be pairwise stable if it is individually rational and is not blocked by any buyer-seller pair. A 4-tuple $(X; p, q, r)$ of a matching X and a feasible price vector p is said to be a *pairwise-stable outcome* if the following two conditions are satisfied:

(p1) $q \geq \mathbf{0}$ and $r \geq \mathbf{0}$,

(p2) $f_{ij}(c) \leq q_i$ or $f_{ji}(-c) \leq r_j$ for all $c \in [\underline{p}_{ij}, \bar{p}_{ij}]_{\mathbb{Z}}$ and for all $(i, j) \in E^5$,

²The notation \mathbb{Z} stand for set of integers and notation \mathbb{R} stand for set of real numbers. The notation \mathbb{Z}^E stands for integer lattice whose points are indexed by E .

³For any two vectors $x \in \mathbb{Z}^E$ and $y \in \mathbb{Z}^E$, we say that $x \leq y$ if $x_{ij} \leq y_{ij}$ for all $(i, j) \in E$.

⁴By strictly increasing function we mean that for $x > y$ implies $f(x) > f(y)$.

⁵we define $[x, y]_{\mathbb{Z}} = \{a \in \mathbb{Z} \mid x \leq a \leq y\}$.

where $(q, r) \in \mathbb{R}^U \times \mathbb{R}^V$ is defined by

$$q_i = \begin{cases} f_{ij}(p_{ij}) & \text{if } (i, j) \in X \text{ for some } j \in V \\ 0 & \text{otherwise} \end{cases} \quad (i \in U), \quad (4.1)$$

$$r_j = \begin{cases} f_{ji}(-p_{ij}) & \text{if } (i, j) \in X \text{ for some } i \in U \\ 0 & \text{otherwise} \end{cases} \quad (j \in V). \quad (4.2)$$

Condition (p1) says that the matching X is individually rational. Condition (p2) means $(X; p, q, r)$ is not blocked by any buyer-seller pair. A matching X is said to be *pairwise-stable* if $(X; p, q, r)$ is pairwise-stable.

To show the existence of pairwise-stable outcome in the model defined in Section 3, we first need to calculate price vector p for each buyer-seller pairs. Since prices should be feasible and $p_{ij} \in \mathbb{Z}$ for each $(i, j) \in E^6$, so initially we define it by

$$p_{ij} = \begin{cases} \bar{\pi}_{ij} & \text{if } f_{ji}(-\bar{\pi}_{ij}) \geq 0 \\ \max \left\{ \bar{\pi}_{ij}, \left\lfloor -f_{ji}^{-1}(0) \right\rfloor \right\} & \text{otherwise.} \end{cases} \quad (4.3)$$

Before describing the algorithm mathematically, we define few subsets of set E that help us to find a matching X satisfying condition (p1). Firstly, we define the subset K_0 and T_0 of set E , that contain those buyer-seller pairs from the set E that are not mutually acceptable, as:

$$K_0 = \{(i, j) \in E \mid f_{ji}(-p_{ij}) < 0\}, \quad (4.4)$$

$$T_0 = \{(i, j) \in E \mid f_{ij}(p_{ij}) < 0\}. \quad (4.5)$$

K_0 is the set of all those pairs where buyer is not ready to trade with seller and T_0 is the set of all those pairs where seller is not ready to trade with buyer. Now the set of mutually acceptable buyer-seller pairs is defined as:

$$\tilde{E} = E \setminus \{K_0 \cup T_0\}. \quad (4.6)$$

Define \tilde{q}_i for each $i \in U$, and \tilde{E}_P by (4.7) and (4.8)

$$\tilde{q}_i = \max\{f_{ij}(p_{ij}) \mid (i, j) \in \tilde{E}\} \quad (4.7)$$

and

$$\tilde{E}_P = \{(i, j) \in \tilde{E} \mid f_{ij}(p_{ij}) = \tilde{q}_i\}. \quad (4.8)$$

The maximum over an empty set is taken to be zero by definition. Here the set \tilde{E}_P contains those buyer-seller pairs which are mutually acceptable and the buyer is most preferred for seller out of all acceptable buyers. We define a subset \hat{E}_P of \tilde{E}_P by:

$$\hat{E}_P = \{(i, j) \in \tilde{E}_P \mid f_{ji}(-p_{ij}) \geq r_j\}. \quad (4.9)$$

Initially, since $r = \mathbf{0}$, \hat{E}_P will coincide with \tilde{E}_P . However, in the further iterations of the algorithm \hat{E}_P may be a proper subset of \tilde{E}_P .

Since we have no matching X at the start of the algorithm, so consider $\tilde{V} = \emptyset$, where \tilde{V} denotes the set of matched buyers in X , that is,

⁶ $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} \mid x \geq n\}.$

$$\tilde{V} = \{j \in V \mid j \text{ is matched in } X\}. \quad (4.10)$$

If $\tilde{V} = \emptyset$, then there is no matched buyer in matching X . At each step in the algorithm, the matching X in the bipartite graph $(U, V; \hat{E}_P)$ must satisfy the following conditions:

(s1) X matches all members of \tilde{V} ,

(s2) X maximizes $\sum_{(i,j) \in X} f_{ji}(-p_{ij})$ among the matchings that satisfy (s1).

Up to this point the outcome $(X; p, q, r)$ obviously satisfies the condition (p1). To satisfy the condition (p2), we define the set K of all those buyer-seller pairs that are mutually acceptable and the buyer is most preferred to seller but the seller is unmatched in X by

$$K = \{(i, j) \in \tilde{E}_P \mid i \text{ is unmatched in } X\}. \quad (4.11)$$

Lemma 4.1. *If $K = \emptyset$, then matching X is stable.*

Proof. We know that a stable matching satisfies conditions (p1) and (p2). By definition $X \subseteq \tilde{E}$ thus (p1) holds true. Suppose that $K = \emptyset$ and on contrary suppose that (p2) does not hold true. This means that for some $(i, j) \in E$ there exists $c \in [\underline{\pi}_{ij}, \bar{\pi}_{ij}]$ such that $f_{ji}(-c) > r_j$ and $f_{ij}(c) > q_i$. Initially, $r = \mathbf{0}$ and $f_{ji}(-p_{ij}) \geq 0$ for $(i, j) \in E$, therefore, $p_{ij} \geq c$, by (4.3). This means that $f_{ij}(p_{ij}) \geq f_{ij}(c) > q_i$. But $K = \emptyset$ implies that $f_{ij}(p_{ij}) < \tilde{q}_i = q_i$, which is a contradiction. This proves the assertion. \square

If $K = \emptyset$ then there is no need to modify price vector p and define further sets but if K is not empty then we will modify price vector, by preserving condition (p1). The new price vector must also be feasible, that is, $\underline{\pi}_{ij} \leq \tilde{p}_{ij} \leq \bar{\pi}_{ij}$ for each $(i, j) \in E$. Since we are considering strictly increasing functions, therefore, we can find a real number $m_{ij}^* \in \mathbb{R}^{++}$ for each $(i, j) \in K$, to modify price vector p , such that

$$f_{ji}(-(p_{ij} - m_{ij}^*)) = r_j. \quad (4.12)$$

Since we are dealing with discrete prices so we will define an integer m_{ij} as follows:

$$m_{ij} = \max \{1, \lceil m_{ij}^* \rceil\}. \quad (4.13)$$

Now, we have

$$f_{ji}(-(p_{ij} - m_{ij})) \geq r_j, \quad (4.14)$$

where $p_{ij} - m_{ij}$ is an integer and m_{ij} is the minimum positive integer that satisfies the above condition.

This means that

$$f_{ji}(-(p_{ij} - (m_{ij} - 1))) \leq r_j.$$

Here the integer m_{ij} for each $(i, j) \in K$ helps us in finding the new price vector such that condition (p2) also satisfies. Now we define a subset L of K that contains those pairs from the set K for which modified price does not remain feasible.

$$L = \{(i, j) \in K \mid p_{ij} - m_{ij} < \underline{\pi}_{ij}\}. \quad (4.15)$$

The modified price vector \tilde{p} must also be feasible and is defined by:

$$\tilde{p}_{ij} := \begin{cases} \max\{\underline{p}_{ij}, p_{ij} - m_{ij}\} & \text{if } (i, j) \in K \\ p_{ij} & \text{otherwise} \end{cases} \quad (i, j) \in E. \quad (4.16)$$

We also define a subset \tilde{T}_0 of K by:

$$\tilde{T}_0 := \{(i, j) \in K \mid f_{ij}(\tilde{p}_{ij}) < 0\}. \quad (4.17)$$

Remark: Throughout in the algorithm, our modified price vector will be decreasing and the size of matching X will be increasing. Also, the participants will change their preferences according to new price vector.

5 An Algorithm for Finding a Pairwise Stability

In this section, we propose an algorithm for finding a pairwise stable outcome for the model described in Section 3.

Input: Two disjoint and finite sets U and V , the set of ordered pairs $E = U \times V$, price vector $p \in \mathbb{Z}^E$, two vectors $\underline{p} \in \mathbb{Z}^E$ and $\bar{p} \in \mathbb{Z}^E$ where $\underline{p} \leq \bar{p}$, general increasing functions .

Output: Vectors $(q, r) \in \mathbb{R}^U \times \mathbb{R}^V$, and $p \in \mathbb{Z}^E$ must satisfy (p1) and (p2).

Step 0: Put $\tilde{V} = \emptyset$ and $r = \mathbf{0}$. Initially define p , K_0 , T_0 , \tilde{E} , \tilde{q} , \tilde{E}_P and \hat{E}_P by (4.3)–(4.9), respectively and find a matching X in the bipartite graph $(U, V; \hat{E}_P)$ satisfying (s1) and (s2). Define r , \tilde{V} and K by (4.2), (4.10) and (4.11), respectively.

Step 1: If $K = \emptyset$ then define q by (4.1) and stop. Otherwise go to Step 2.

Step 2: For each $(i, j) \in K$ calculate m_{ij} by (4.13) and new price vector \tilde{p} by (4.16). Define L and \tilde{T}_0 by (4.15) and (4.17), respectively and update T_0 by $T_0 := T_0 \cup \tilde{T}_0$ and K_0 by $K_0 := K_0 \cup L$.

Step 3: Replace price vector p by \tilde{p} and modify \tilde{E} by:

$$\tilde{E} := \tilde{E} \setminus \{K_0 \cup T_0\}. \quad (5.18)$$

Again define \tilde{q} by (4.7) and modify \tilde{E}_P , \hat{E}_P by (4.8) and (4.9) respectively, for the updated p and \tilde{E} . Find a matching X in the bipartite graph $(U, V; \hat{E}_P)$ that satisfies the conditions (s1) and (s2). Again define r , \tilde{V} and K by (4.2), (4.10) and (4.11), respectively. Go to Step 1.

6 Existence of Pairwise Stability

In this section, we will show the existence of pairwise stability for this model. For this purpose, we will show that the algorithm we have proposed terminates and at termination it outputs a stable matching. We will also give some other important results about the model and the algorithm.

We will add prefixes *(old)** and *(new)** to sets/vectors/integers before and after update, respectively, in any iteration of the algorithm. The key result is Lemma 6.2 which will be proved here using the assumption defined in equation (4.13).

Lemma 6.1. *There exists a matching X in the bipartite graph $(U, V; \widehat{E}_P)$ that satisfy condition (s1) and (s2) in each iteration of the algorithm at Step 3.*

Proof. The proof of the lemma is equivalent to show that $(old)X \subseteq (new)\widehat{E}_P$ at Step 3, in each iteration. In each iteration at Step 2 and at Step 3, we update vector p and \widehat{E} by (4.16) and (5.18), respectively. As clear from (4.16) and (5.18), these modifications are done for elements or/and subsets of K . As $K \cap (old)X = \emptyset$, therefore, $(old)X \subseteq (new)\widehat{E}_P$. \square

The following lemma represents the significance of m_{ij} for each $(i, j) \in K$ and explains that updated price is the maximum price at which $(i, j) \in K$ can match.

Lemma 6.2. *In each iteration of the algorithm at Step 3, we have $f_{ji}(-(p_{ij} - m_{ij})) \geq r_j$ for each $(i, j) \in K$. Furthermore, if $f_{ji}(-(p_{ij} - m_{ij})) > r_j$ for some $(i, j) \in K$ then $p_{ij} - m_{ij}$ is the maximum integer for which this inequality holds.*

Proof. Let $(i, j) \in K$ this means that $f_{ji}(-(old)p_{ij}) \leq r_j$. At Step 2 we calculated an integer m_{ij} by (4.13) for each $(i, j) \in K$ with following property

$$f_{ji}(-(old)p_{ij} - m_{ij}) \geq r_j.$$

This proves the first part of the assertion.

Next, we prove the second part of the lemma that if $f_{ji}(-(old)p_{ij} - m_{ij}) > r_j$ then $(old)p_{ij} - m_{ij}$ is the maximum integer for which this holds. This can be proven by showing that m_{ij} is minimum positive integer for which $f_{ji}(-(old)p_{ij} - m_{ij}) > r_j$ holds.

By (4.13), we have $m_{ij} \geq 1$. First we consider the case when $m_{ij}^* \leq 1$, that is, $m_{ij} = 1$ by (4.13). For this case the result holds trivially as $m_{ij} = 1$ is minimum positive integer. Now, consider when $m_{ij}^* > 1$. We m_{ij}^* is a real number for which we have

$$r_j = f_{ji}(-(old)p_{ij} + m_{ij}^*)$$

As we are dealing with strictly increasing function, therefore, for any real number $\delta > 0$, we have

$$f_{ji}(-(old)p_{ij} - (m_{ij}^* + \delta)) > r_j > f_{ji}(-(old)p_{ij} - (m_{ij}^* - \delta)). \quad (6.19)$$

Since

$$m_{ij} = \lceil m_{ij}^* \rceil \geq m_{ij}^*. \quad (6.20)$$

By (6.19), m_{ij}^* is minimum positive real number for which $r_j < f_{ji}(-(old)p_{ij} + m_{ij}^*)$ and by (6.20), m_{ij} is minimum positive integer for which $f_{ji}(-(p_{ij} - m_{ij})) > r_j$. Thus $(old)p_{ij} - m_{ij}$ is the maximum integer for which $f_{ji}(-(p_{ij} - m_{ij})) > r_j$ holds. \square

For $(i, j) \in K$, we update price vector by (4.16). There is a possibility that $(new)p_{ij}$ does not remain feasible, that is, $p_{ij} < \underline{\pi}_{ij}$. To maintain the feasibility in such cases we have the following result.

Lemma 6.3. *For each $(i, j) \in L$ we have $(new)p_{ij} = \underline{\pi}_{ij}$ and $f_{ji}(-(new)p_{ij}) \leq (old)r_j$, where L is defined at Step 2.*

The proof of the Lemma 6.3 follows by using (4.15) and (4.16).

The following lemma describes the important features of our algorithm. The results of these lemma will be used to show that the algorithm will terminate after finite number of iterations.

Lemma 6.4. *In each iteration of the algorithm, following hold:*

- (i) *If $L \neq \emptyset$ or $\tilde{T}_0 \neq \emptyset$ at Step 2 then \tilde{E} reduces at Step 3. Otherwise \tilde{E} will remain the same.*
- (ii) *The vector p decreases or remains same. In particular, if $K \setminus \{L \cup \tilde{T}_0\} \neq \emptyset$ at Step 2 then p_{ij} decreases at Step 3 for all $(i, j) \in K \setminus \{L \cup \tilde{T}_0\}$.*
- (iii) *The vector r increases or remains same.*

Proof. (i) At Step 0, \tilde{E} is given by (4.6) and it is updated by (5.18), at Step 3. At Step 2 we updated $K_0 = K_0 \cup L$ and $T_0 = T_0 \cup \tilde{T}_0$. According to (5.18), \tilde{E} will reduce if $L \neq \emptyset$ or $\tilde{T}_0 \neq \emptyset$ at Step 2. If both L and \tilde{T}_0 are empty the \tilde{E} will remain unchanged by (5.18).

(ii) Initially, p is set by (4.3) and in each iteration it is updated by (4.16). It is easy to see that for $(i, j) \in K$, $\tilde{p}_{ij} \leq p_{ij}$. Here the equality may hold for $(i, j) \in L \cup \tilde{T}_0$.

(iii) At the start of the algorithm we set $r = \mathbf{0}$. We modified r by (4.2) afterwards. In each iteration, matching X satisfies condition (s1) this means that $(old)\tilde{V} \subseteq (new)\tilde{V}$. Also, $(new)p \leq (old)p$ by part (ii) of Lemma 6.4. Thus $(new)r_j = f_{ji}(-(new)p_{ij}) \geq (old)r_j$, for $j \in (old)\tilde{V}$, as matching X also satisfies (s2). Moreover, $(old)r_j = (new)r_j = 0$ for each $j \in V \setminus \tilde{V}$. Therefore, vector r either remains the same or increases. □

To show that our algorithm produces a stable matching is not possible without proving that our algorithm will terminate after some iterations.

Theorem 6.5. *The algorithm terminates after finite number of iterations.*

Proof. Termination of the algorithm depends upon set of mutually acceptable pairs and price vector p . By the Lemma 6.4 part (i), \tilde{E} reduces when either $L \neq \emptyset$ or $T_0 \neq \emptyset$ or remains the same. This case is possible at most $|E|$ times.

If $L = T_0 = \emptyset$ then, by part (ii) of Lemma 6.4, p_{ij} decreases for each $(i, j) \in K$. Otherwise, p remains unchanged. As we know that p is bounded and discrete, therefore, it can be decreased a finite number of time. This proves that in either case our algorithm terminates after a finite number of iterations. □

This is the most important result which establishes the existence of pairwise stability for our model.

Theorem 6.6. *The outcome $(X; p, q, r)$ must satisfies the condition (p1) and (p2) if algorithm terminates.*

Proof. We know that $X \subseteq \tilde{E}$. Initially \tilde{E} is defined by (4.6) and afterwards it is updated by (5.18) at Step 3 in each iteration. Thus $f_{ij}(p_{ij})$ and $f_{ji}(-p_{ij})$ are non-negative for all $(i, j) \in \tilde{E}$. Therefore, $f_{ij}(p_{ij}) \geq 0$ and $f_{ji}(-p_{ij}) \geq 0$ for all $(i, j) \in X$. This shows that the X satisfies (p1) at termination.

On contrary to (p2), assume that there exist $\alpha \in [\underline{\pi}_{ij}, \bar{\pi}_{ij}]$ and $(i, j) \in E$ such that

$$f_{ij}(\alpha) > q_i \quad \text{and} \quad f_{ji}(-\alpha) > r_j.$$

If we take $p_{ij} < \alpha$ it yields $f_{ji}(-p_{ij}) > f_{ji}(-\alpha) > r_j$. But according to Lemma 6.2, p_{ij} is the maximum integer for which this inequality holds. Thus $p_{ij} < \alpha$ is not possible. Now consider that $p_{ij} \geq \alpha$, which implies that

$$f_{ij}(p_{ij}) \geq f_{ij}(\alpha) > q_i. \quad (6.21)$$

However, at termination we have $K = \emptyset$ means that $(i, j) \notin K$ and since (i, j) are not matched, therefore, $f_{ij}(p_{ij}) < \tilde{q}_i = q_i$. A contradiction to (6.21). Thus (p2) holds when the algorithm terminates. \square

7 Conclusion

This paper presents a matching model where money is given in integers. The preferences of participants are represented by general increasing utility functions. Ali and Farooq [1] is a special case of our model. We have given a constructive proof for the existence of a pairwise stable outcome in our model. As a future work it is important to consider problems concerning the structures of pairwise stable outcomes in our model. It is well-known that stable matchings forms a lattice. A similar approach can be found in article [4] by Farooq *et al.* It would be worthwhile to prove the existence of stable outcome for many-to-many model with such valuation functions by using the same mathematical apparatus. Further, the complexity of our algorithm may depend on the length of $[\underline{\pi}, \bar{\pi}]$. An interesting problem may be to devise an algorithm with polynomial complexity in the number of participants.

References

- [1] Ali Y, Farooq R. Pairwise stability in a two-sided matching market with indivisible goods and money. J Oper Res Soc Jpn 2011; 54: 1–11.
- [2] Eriksson K, Karlander J. Stable matching in a common generalization of the marriage and assignment models. Discrete Math 2000; 217: 135–156.
- [3] Farooq R. A polynomial-time algorithm for a stable matching problem with linear valuations and bounded side payments. Jpn J Ind Appl Math 2008; 25: 83–98.
- [4] Farooq R, Ikebe Y. T, Tamura A. On a labor allocation model with possible bounded salaries. J OPER RES SOC JPN 2008; 51: No. 2, 136–154.
- [5] Gale D, Shapley LS. College admissions and the stability of marriage. Amer Math Monthly 1962; 69: 9–15.
- [6] Gusfield D, Irving RW. The Stable Marriage Problem Structure and Algorithms. Cambridge, US: MIT Press, 1989.
- [7] Roth A, Sotomayor M. Two-sided matching. A study in game-theoretic modeling and analysis. Econ Soc Monogr, Cambridge University Press, 1990.
- [8] Shapley LS, Shubik M. The assignment game I: The core. Internat J Game Theory 1972; 1: 111–130.

- [9] Sotomayor M. Existence of stable outcomes and the lattice property for a unified matching market. *Math Social Sci* 2000; 39: 119–132.